Additional rules for vector calculations

Add-on for Wednesday, September 11

Components and unit vector

Unit vectors:

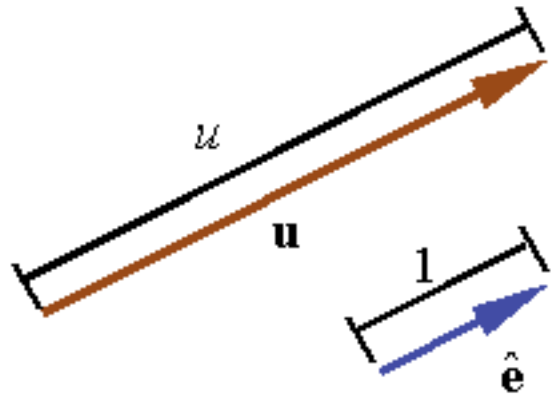
A unit vector is a vector of unit length. A unit vector is sometimes denoted by replacing the arrow on a vector with a "^" or just adding a "^" on a boldfaced character (i.e.,). Therefore,

$|\hat{\mathbf{e}}| = 1$ Any vector can be made into a unit vector by dividing it by its length.

$$\hat{\mathbf{e}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

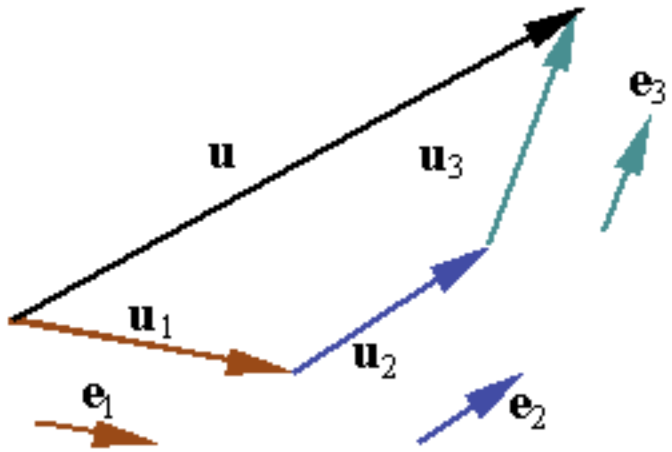
Any vector can be fully represented by providing its magnitude and a unit vector along its direction.

$$\mathbf{u} = u\hat{\mathbf{e}}$$



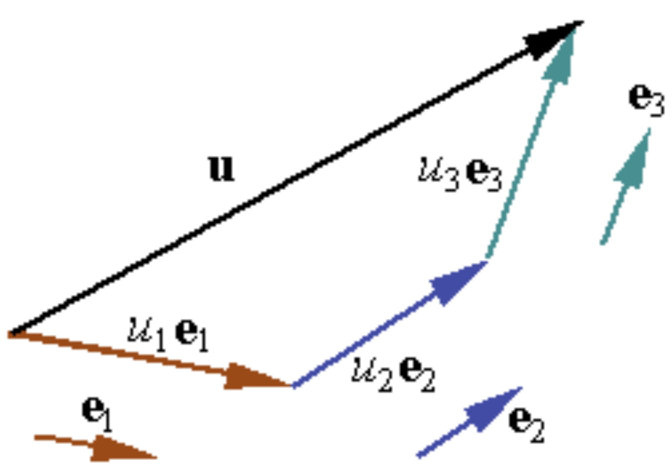
Base vectors are a set of vectors selected as a base to represent all other vectors. The idea is to construct each vector from the addition of vectors along the base directions. For example, the vector in the figure can be written as the sum of the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, each along the direction of one of the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, so that

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$



Each one of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ is parallel to one of the base vectors and can be written as scalar multiple of that base. Let u_1, u_2, and u_3 denote these scalar multipliers such that one has

$$\mathbf{u}_1 = u_1\mathbf{e}_1$$
$$\mathbf{u}_2 = u_2\mathbf{e}_2$$
$$\mathbf{u}_3 = u_3\mathbf{e}_3$$

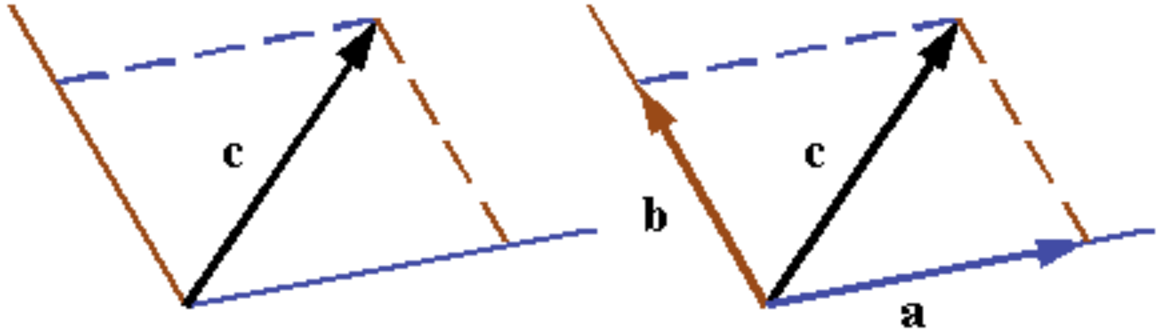


The original vector $\mathbf{u}$ can now be written as:

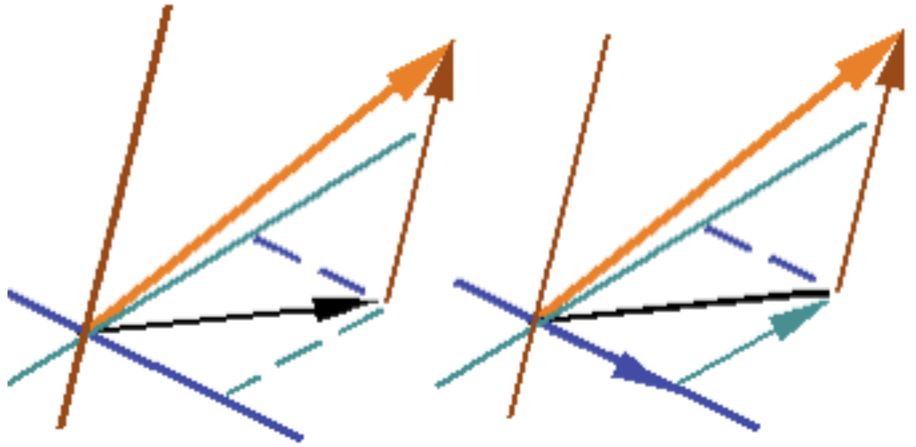
$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

The scalar multipliers u_1, u_2, and u_3 are known as the components of **u** in the base described by the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. If the base vectors are unit vectors, then the components represent the lengths, respectively, of the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. If the base vectors are unit vectors and are mutually orthogonal, then the base is known as an orthonormal, Euclidean, or Cartesian base.

A vector can be resolved along any two directions in a plane containing it. The figure shows how the parallelogram rule is used to construct vectors **a** and **b** that add up to **c**.



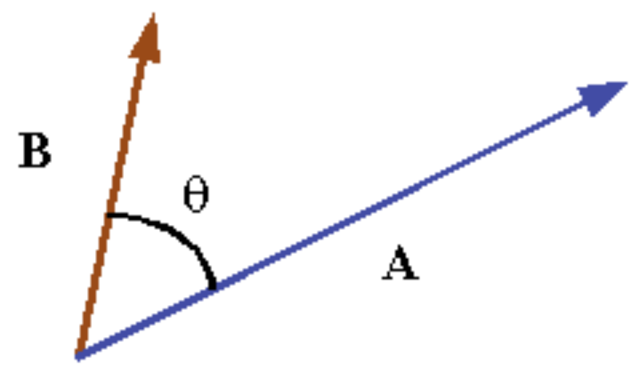
In three dimensions, a vector can be resolved along any three non-coplanar lines. The figure shows how a vector can be resolved along the three directions by first finding a vector in the plane of two of the directions and then resolving this new vector along the two directions in the plane.



Scalar (dot) product

The dot product is denoted by "" between two vectors. The dot product of vectors **A** and **B** results in a scalar given by the relation

$$\mathbf{A} \circ \mathbf{B} = AB\cos(\theta)$$



where is the angle between the two vectors. Order is not important in the dot product as can be seen by the dot products definition. As a result one gets

$$\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$$

The dot product has the following properties.

$$a(\mathbf{b} \circ \mathbf{c}) = (a\mathbf{b}) \circ \mathbf{c} = \mathbf{b} \circ (a\mathbf{c})$$

$$\mathbf{a} \circ (\mathbf{b} + \mathbf{c}) = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \circ \mathbf{c}$$

Since the cosine of 90^{o} is zero, the dot product of two orthogonal vectors will result in zero.

Since the angle between a vector and itself is zero, and the cosine of zero is one, the magnitude of a vector can be written in terms of the dot product using the rule

$$\mathbf{A} \circ \mathbf{A} = A^2$$

Rectangular coordinates:

When working with vectors represented in a rectangular coordinate system by the components

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

then the dot product can be evaluated from the relation

$$\mathbf{A} \circ \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

This can be verified by direct multiplication of the vectors and noting that due to the orthogonality of the base vectors of a rectangular system one has

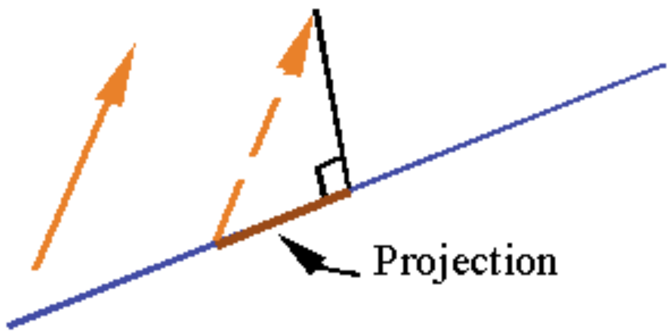
$$\hat{\mathbf{i}} \circ \hat{\mathbf{j}} = 0$$

$$\hat{\mathbf{i}} \circ \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{j}} \circ \hat{\mathbf{k}} = 0$$

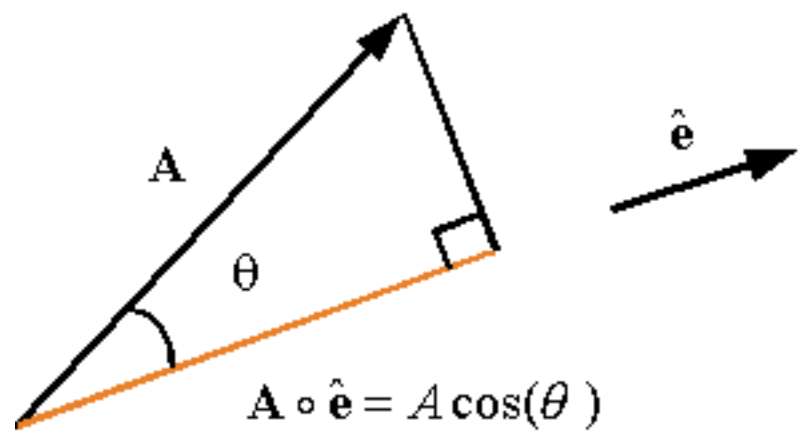
Projection of a vector onto a line:

The orthogonal projection of a vector along a line is obtained by moving one end of the vector onto the line and dropping a perpendicular onto the line from the other end of the vector. The resulting segment on the line is the vector's orthogonal projection or simply its projection.



The scalar projection of vector **A** along the unit vector is the length of the orthogonal projection **A** along a line parallel to , and can be evaluated using the dot product. The relation for the projection is

$$\text{Scalar projection of } \mathbf{A} \text{ along } \hat{\mathbf{e}} = \mathbf{A} \circ \hat{\mathbf{e}}$$



The vector projection of **A** along the unit vector simply multiplies the scalar projection by the unit vector to get a vector along . This gives the relation

$$\text{Vector projection of } \mathbf{A} \text{ along } \hat{\mathbf{e}} = (\mathbf{A} \circ \hat{\mathbf{e}})\hat{\mathbf{e}}$$